

Statistical mechanics of bipartite z -matchings

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Abstract – The matching problem has a large variety of applications including the allocation of competitive resources and network controllability. The statistical mechanics approach based on the cavity method has shown to be exact in characterising this combinatorial problem on locally tree-like networks. Here we use the cavity method to solve the many-to-one bipartite z -matching problem that can be considered to be a model for the characterisation of the capacity of user-server networks such as wireless communication networks. Finally we study the phase diagram of the model defined in network ensembles.

Introduction. – In network science [1–4] there is increasing interest in combinatorial optimisation problems and message-passing algorithms applied to processes as different as control [5–7], percolation [8, 9], percolation on multilayer networks [4, 10–14] or epidemic spreading [15–18]. On one hand, this surge of interest is motivated by the efficiency of using statistical mechanics approach [19, 20] in solving combinatorial optimisation problems. On the other hand, it comes as a consequence of the vast realm of applications of these problems and their generalisations. In this paper we characterise the statistical mechanics of a generalised matching problem called z -matching that can be interpreted as a model in a system with limited resources.

On an undirected network the matching problem consists of finding the maximum subset of links of the network (the matched links) such that each node is adjacent to at most one link from that subset. This problem has attracted large interest from combinatorics, probability and the computer science communities [21–23]. For this problem the statistical mechanics approach [24–26] is very useful and in particular the Belief Propagation algorithm [27–30] provides the exact solution as long as the network is locally tree-like.

The matching problem and its generalisations have a variety of applications ranging from wireless networks to network controllability. The generalisation of the matching problem on directed networks has recently been shown [5]

to characterise the network controllability as it identifies the set of driver nodes of the network. Since the matching problem on directed locally tree-like networks is exactly solvable using statistical mechanics methods, this result has opened new perspectives in network controllability. In particular, it has allowed to relate the directed network dynamical properties (controllability) to its structural ones (the properties of its maximum matching). Interestingly in this context it has been shown [6] that the key structural property characterising the matching (and hence the network controllability) is the fraction of nodes with low in and out degrees. These results have also recently been extended to multilayer network controllability by considering the relevant extension of the maximum matching problem to a multilayer network maximum matching problem [7].

More traditionally the matching problem has been defined in spatial networks where nodes have specific positions, or more generally in networks in which each pair of nodes is associated with a distance, the so called marriage problem [31–33]. In this setting the generalised matching problem aims at finding the maximum matching that minimises the overall distances between the nodes. This specific model defined over a bipartite network has a variety of applications [31] in finite resource allocation problems where some providers of service want to optimise the user satisfaction and their own profit.

In this paper, we are focusing on another important variation of the matching problem called the many-to-one

z -matching on bipartite networks. In this case we consider a bipartite network in which one set of nodes can be matched at most to one link and the other set of nodes can be matched to at most $z > 1$ links. The many-to-one z -matching problem on bipartite networks has recently become particularly relevant for characterising wireless communication networks [34,35] in which one has two different sets of nodes – users and towers. Each tower provides the wireless connection, but can only serve up to z users. The *maximum capacity* of the network is given by the largest number of users that can be served at a time. Here, we determine the Belief Propagation equations to characterise the maximum capacity of any locally tree-like bipartite network and we evaluate the capacity of network ensembles with given degree distribution. Our analysis is based on the use of the Belief Propagation algorithm in the zero-temperature limit. In this way we extend the existing statistical mechanics treatment of the one-to-one maximum matching problem of simple, directed and multilayer networks to the many-to-one matching problem.

The maximum z -matching problem. – Let us consider a bipartite network formed by N users $i = 1, \dots, N$ and M towers $\alpha = 1, \dots, M$ where we assume for simplicity that each tower is connected to at least z users. A z -matching is the subset of the set of edges such that each user is adjacent to at most one and each tower is adjacent to at most z edges from the subset. In other words, each user communicates with at most one neighbouring tower and each tower serves at most z neighbouring users. The *size* of a z -matching is given by the number of its edges and the *maximum capacity* of the network is given by the size of the largest possible z -matching. In order to treat our problem from the statistical mechanics point of view, for each linked pair (i, α) (i.e. a pair user-tower connected by an edge in the network), and a given z -matching, we introduce variables $s_{i\alpha} \in \{1, 0\}$ such that $s_{i\alpha} = 1$ if the edge is included in the z -matching, and $s_{i\alpha} = 0$ otherwise. A z -matching reduces to an assignment $\{s_{i\alpha}\}$ that satisfies the conditions

$$\begin{aligned} \sum_{\alpha \in N(i)} s_{i\alpha} &\leq 1, \\ \sum_{i \in N(\alpha)} s_{i\alpha} &\leq z, \end{aligned} \quad (1)$$

where $N(i)$ denotes the set of neighbours of a node i and $N(\alpha)$ denotes the set of neighbours of a tower α . Let us define the *energy* E and the *capacity* C of the z -matching as

$$\begin{aligned} E &= \sum_{i=1}^N E_i + \sum_{\alpha=1}^M E_\alpha, \\ C &= \sum_{i=1}^N (1 - E_i), \end{aligned} \quad (2)$$

with E_i and E_α given by

$$\begin{aligned} E_i &= 1 - \sum_{\alpha \in N(i)} s_{i\alpha}, \\ E_\alpha &= z - \sum_{i \in N(\alpha)} s_{i\alpha}. \end{aligned} \quad (3)$$

Therefore, the capacity corresponds to the number of users with one matched link. Having in mind that $\sum_{i=1}^N \sum_{\alpha \in N(i)} s_{i\alpha} = \sum_{\alpha=1}^M \sum_{i \in N(\alpha)} s_{i\alpha}$, we have the following simple relationship

$$E = (zM + N) - 2C. \quad (4)$$

The problem of finding a maximum capacity C of the z -matching translates then to the problem of investigating allowed configurations $\{s_{i\alpha}\}$ for the z -matching which minimise the energy E .

Here we associate to each possible z -matching of the network the Gibbs measure

$$\hat{P}(\{s_{i\alpha}\}) = \frac{e^{-\beta E}}{Z} \prod_{i=1}^N \theta(1 - \sum_{\alpha \in N(i)} s_{i\alpha}) \cdot \prod_{\alpha=1}^M \theta(z - \sum_{i \in N(\alpha)} s_{i\alpha}),$$

where $\beta > 0$ denotes the inverse temperature, Z is a normalisation constant and $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. The free-energy of the problem $F(\beta)$ is defined as

$$\beta F(\beta) = -\ln Z, \quad (5)$$

and the energy E is given by

$$E = \frac{\partial[\beta F(\beta)]}{\partial \beta}. \quad (6)$$

In order to characterise the maximum capacity of a network, or equivalently its minimum energy, we are interested in the limit when $\beta \rightarrow \infty$.

The Belief Propagation solution on a single network. –

The Belief Propagation equations. On a locally tree like bipartite networks, such as a random bipartite network, the Gibbs measure given by Eq. (5) can be found in the Bethe approximation using the Belief Propagation (BP) algorithm [19]. The BP algorithm expresses the Gibbs measure in terms of messages. Here we distinguish between two types of messages: those that the users send to neighbouring towers $P_{i \rightarrow \alpha}(s_{i\alpha})$, and those that the towers send to neighbouring users $P_{\alpha \rightarrow i}(s_{i\alpha})$. For a user i and its neighbour α , $P_{i \rightarrow \alpha}(1)$ denotes the probability that i says to α that it believes that their edge should be included to the z -matching. Similarly, $P_{\alpha \rightarrow i}(1)$ denotes the probability that α informs i that their edge should be matched. Then, for $s_{i\alpha} \in \{0, 1\}$ the Belief Propagation messages are given by

$$P_{i \rightarrow \alpha}(s_{i\alpha}) = \frac{1}{C_{i \rightarrow \alpha}} \sum_{\mathbf{s}_{i \setminus s_{i\alpha}}} e^{-\beta E_i} \theta(1 - \sum_{\gamma \in N(i)} s_{i\gamma})$$

$$\begin{aligned}
 P_{\alpha \rightarrow i}(s_{i\alpha}) &= \frac{1}{\mathcal{C}_{\alpha \rightarrow i}} \sum_{\mathbf{s}_{\alpha} \setminus s_{i\alpha}} e^{-\beta E_{\alpha}} \theta(z - \sum_{j \in N(\alpha)} s_{j\alpha}) \\
 &\times \prod_{\gamma \in N(i) \setminus \alpha} P_{\gamma \rightarrow i}(s_{i\gamma}), \quad (7)
 \end{aligned}$$

where $\mathbf{s}_i = \{s_{i\alpha} | \alpha \in N(i)\}$, $\mathbf{s}_{\alpha} = \{s_{i\alpha} | i \in N(\alpha)\}$, and $\mathcal{C}_{i \rightarrow \alpha}$, $\mathcal{C}_{\alpha \rightarrow i}$, represent normalisation constants. The messages $P_{i \rightarrow \alpha}(s_{i\alpha})$, $P_{\alpha \rightarrow i}(s_{i\alpha})$ can be parametrized by the cavity fields $h_{i \rightarrow \alpha}$, $\hat{h}_{\alpha \rightarrow i}$ in the following way

$$\begin{aligned}
 P_{i \rightarrow \alpha}(s_{i\alpha}) &= \frac{e^{\beta h_{i \rightarrow \alpha} s_{i\alpha}}}{1 + e^{\beta h_{i \rightarrow \alpha}}}, \\
 P_{\alpha \rightarrow i}(s_{i\alpha}) &= \frac{e^{\beta \hat{h}_{\alpha \rightarrow i} s_{i\alpha}}}{1 + e^{\beta \hat{h}_{\alpha \rightarrow i}}}. \quad (8)
 \end{aligned}$$

By using this parametrization of the messages the BP equations can be written explicitly as

$$\begin{aligned}
 P_{i \rightarrow \alpha}(0) &= \frac{1}{\mathcal{C}_{i \rightarrow \alpha}} \left(e^{-\beta} + \sum_{\gamma \in N(i) \setminus \alpha} e^{\beta \hat{h}_{\gamma \rightarrow i}} \right) \\
 &\times \prod_{\gamma \in N(i) \setminus \alpha} P_{\gamma \rightarrow i}(0), \\
 P_{i \rightarrow \alpha}(1) &= \frac{1}{\mathcal{C}_{i \rightarrow \alpha}} \prod_{\gamma \in N(i) \setminus \alpha} P_{\gamma \rightarrow i}(1), \\
 P_{\alpha \rightarrow i}(0) &= \frac{1}{\mathcal{C}_{\alpha \rightarrow i}} \\
 &\times \left(\sum_{p=0}^z e^{-\beta(z-p)} \sum_{j_1, \dots, j_p} e^{\sum_{m=0}^p \beta h_{j_m \rightarrow \alpha}} \right) \\
 &\times \prod_{j \in N(\alpha) \setminus i} P_{j \rightarrow \alpha}(0), \\
 P_{\alpha \rightarrow i}(1) &= \frac{1}{\mathcal{C}_{\alpha \rightarrow i}} \\
 &\times \left(\sum_{p=0}^{z-1} e^{-\beta(z-1-p)} \sum_{j_1, \dots, j_p} e^{\sum_{m=0}^p \beta h_{j_m \rightarrow \alpha}} \right) \\
 &\times \prod_{j \in N(\alpha) \setminus i} P_{j \rightarrow \alpha}(1). \quad (9)
 \end{aligned}$$

Using Eqs. (9) and Eqs. (8), the BP Eqs. (7) can find closed expression for the cavity fields given by

$$\begin{aligned}
 e^{-\beta h_{i \rightarrow \alpha}} &= e^{-\beta} + \sum_{\gamma \in N(i) \setminus \alpha} e^{\beta \hat{h}_{\gamma \rightarrow i}}, \quad (10) \\
 e^{-\beta \hat{h}_{\alpha \rightarrow i}} &= e^{-\beta} \\
 &+ \frac{\sum_{j_1, \dots, j_l, \dots, j_z | j_l \in N(\alpha) \setminus i} e^{\sum_{l=0}^z \beta h_{j_l \rightarrow \alpha}}}{\sum_{p=0}^{z-1} e^{-\beta(z-p-1)} \cdot \sum_{j_1, \dots, j_l, \dots, j_p | j_l \in N(\alpha) \setminus i} e^{\sum_{l=0}^p \beta h_{j_l \rightarrow \alpha}}}.
 \end{aligned}$$

The marginal probabilities. According to the BP algorithm [19], the marginal probability $P_{i\alpha}(s_{i\alpha})$ of the variable $s_{i\alpha}$ associated with the link between i and α is given

by

$$P_{i\alpha}(s_{i\alpha}) = \frac{1}{\mathcal{C}_{i\alpha}} P_{i \rightarrow \alpha}(s_{i\alpha}) P_{\alpha \rightarrow i}(s_{i\alpha}), \quad (11)$$

where $\mathcal{C}_{i\alpha}$ are normalisation constants.

Similarly the marginal probability $P_i(\mathbf{s}_i)$ of the variables $\mathbf{s}_i = \{s_{i\alpha} | \alpha \in N(i)\}$ associated to the links incident to node i and the marginal probability $P_{\alpha}(\mathbf{s}_{\alpha})$ of the variables $\mathbf{s}_{\alpha} = \{s_{i\alpha} | i \in N(\alpha)\}$ associated to links incident to tower α are given by

$$\begin{aligned}
 P_i(\mathbf{s}_i) &= \frac{e^{-\beta E_i}}{\mathcal{C}_i} \theta(1 - \sum_{\alpha \in N(i)} s_{i\alpha}) \prod_{\alpha \in N(i)} P_{\alpha \rightarrow i}(s_{i\alpha}), \\
 P_{\alpha}(\mathbf{s}_{\alpha}) &= \frac{e^{-\beta E_{\alpha}}}{\mathcal{C}_{\alpha}} \theta(z - \sum_{i \in N(\alpha)} s_{i\alpha}) \prod_{i \in N(\alpha)} P_{i \rightarrow \alpha}(s_{i\alpha}). \quad (12)
 \end{aligned}$$

In the Bethe approximation, valid on locally tree-like networks the Gibbs measure given by Eq. (5) is given in terms of the marginals by

$$\hat{P}(\{s_{i\alpha}\}) = \frac{\prod_{i=1}^N P_i(\mathbf{s}_i) \prod_{\alpha=1}^M P_{\alpha}(\mathbf{s}_{\alpha})}{\prod_{(i,\alpha)} P_{i\alpha}(s_{i\alpha})}. \quad (13)$$

Free energy. The free energy of the system can be found by minimising the Gibbs free energy given by

$$\beta F_{Gibbs} = \sum_{\{s_{i\alpha}\}} \hat{P}(\{s_{i\alpha}\}) \ln \left(\frac{\hat{P}(\{s_{i\alpha}\})}{\psi(\{s_{i\alpha}\})} \right), \quad (14)$$

where

$$\psi(\{s_{i\alpha}\}) = e^{-\beta E} \prod_{i=1}^N \theta(1 - \sum_{\alpha \in N(i)} s_{i\alpha}) \cdot \prod_{\alpha=1}^M \theta(z - \sum_{i \in N(\alpha)} s_{i\alpha}). \quad (15)$$

In fact it can easily be shown that the Gibbs free energy is minimised when $\hat{P}(\{s_{i\alpha}\})$ is given by Eq. (5) and that the minimum Gibbs free-energy is equal to the free energy of the problem and takes the value

$$\beta F_{Gibbs} = \beta F(\beta) = -\ln Z. \quad (16)$$

Using the Bethe expression for the Gibbs measure given by Eq. (13) in Eq. (14) we obtain the free energy

$$\begin{aligned}
 \beta F(\beta) &= \sum_{(i,\alpha)} \ln \mathcal{C}_{i\alpha} - \sum_{i=1}^N \ln \mathcal{C}_i - \sum_{\alpha=1}^M \ln \mathcal{C}_{\alpha} \\
 &= \sum_{(i,\alpha)} \ln \left(1 + e^{\beta(h_{i \rightarrow \alpha} + \hat{h}_{\alpha \rightarrow i})} \right) \\
 &- \sum_{i=1}^N \ln \left(e^{-\beta} + \sum_{\gamma \in N(i)} e^{\beta \hat{h}_{\gamma \rightarrow i}} \right) \\
 &- \sum_{\alpha=1}^M \ln \left(\sum_{p=0}^z e^{-\beta(z-p)} \sum_{j_1, \dots, j_l, \dots, j_p | j_l \in N(\alpha)} e^{\beta \sum_{l=0}^p h_{j_l \rightarrow \alpha}} \right). \quad (17)
 \end{aligned}$$

Finally, using Eq. (6) we can express the energy E in terms of the cavity fields as

$$E = \sum_{(i,\alpha)} \frac{e^{\beta(h_{i \rightarrow \alpha} + \hat{h}_{\alpha \rightarrow i})} (h_{i \rightarrow \alpha} + \hat{h}_{\alpha \rightarrow i})}{1 + e^{\beta(h_{i \rightarrow \alpha} + \hat{h}_{\alpha \rightarrow i})}} + \sum_{i=1}^N \frac{e^{-\beta} - \sum_{\gamma \in N(i)} e^{\beta \hat{h}_{\gamma \rightarrow i}} \hat{h}_{\gamma \rightarrow i}}{e^{-\beta} + \sum_{\gamma \in N(i)} e^{\beta \hat{h}_{\gamma \rightarrow i}}} + \sum_{\alpha=1}^M \frac{\sum_{p=0}^z \sum_{j_1, \dots, j_p} e^{-\beta(z-p-\sum_{l=1}^p h_{j_l \rightarrow \alpha})} g(\{h_{\ell \rightarrow \alpha}\})}{\sum_{p=0}^z e^{-\beta(z-p)} \sum_{j_1, \dots, j_p} e^{\beta \sum_{l=1}^p h_{j_l \rightarrow \alpha}}}, \quad (18)$$

where

$$g(\{h_{\ell \rightarrow \alpha}\}) = \left[(z-p) - \theta(p) \sum_{l=1}^p h_{j_l \rightarrow \alpha} \right]. \quad (19)$$

The zero temperature limit. For finding the maximum capacity of a bipartite network we need to investigate the zero temperature limit of the BP equations, i.e. we need to consider the limit $\beta \rightarrow \infty$. In this limit the cavity fields $h_{i\alpha}, \hat{h}_{\alpha \rightarrow i}$ have the support on $\{-1, 0, 1\}$ and the BP equations are

$$h_{i \rightarrow \alpha} = -\max \left[-1, \max_{\gamma \in N(i) \setminus \alpha} \hat{h}_{\gamma \rightarrow i} \right],$$

$$\hat{h}_{\alpha \rightarrow i} = \begin{cases} 1 & \text{if } \sum_{j \in N(\alpha) \setminus i} \delta(-1, h_{j \rightarrow \alpha}) \geq q - z, \\ -1 & \text{if } \sum_{j \in N(\alpha) \setminus i} \delta(1, h_{j \rightarrow \alpha}) \geq z, \\ 0 & \text{otherwise,} \end{cases} \quad (20)$$

where $\delta(x, y)$ denotes the Kronecker delta. Thus, a node i sets a field $h_{i \rightarrow \alpha} = 1$ if all other neighbouring towers set the fields which point to i to -1 ; it sets $h_{i \rightarrow \alpha} = -1$ if at least one other neighbouring tower sends $+1$; and it sets $h_{i \rightarrow \alpha}$ to 0 otherwise, i.e. if at least one other tower sends 0 , and no tower sends $+1$. Similarly, a tower α of degree $q \geq z$ sets a field $h_{\alpha \rightarrow i}$ to 1 if at least $q - z$ of its neighbouring users set fields pointing to α to -1 ; α sets the field $h_{\alpha \rightarrow i}$ to -1 if at least z neighbours set fields pointing to it to $+1$; and otherwise, it sets the field $h_{\alpha \rightarrow i}$ to 0 .

In the case of multiple solutions, the dynamically stable solution that is physical and that minimises the energy E which can be expressed as

$$E = \sum_{(i,\alpha)} \max \left[0, h_{i \rightarrow \alpha} + \hat{h}_{\alpha \rightarrow i} \right] - \sum_{i=1}^N \max \left[-1, \max_{\gamma \in N(i)} \hat{h}_{\gamma \rightarrow i} \right] - \sum_{\alpha=1}^M \max \left[-z, \max_{\{j_l | j_l \in N(\alpha)\}_{l \leq z}} \sum_{l=1}^z h_{j_l \rightarrow \alpha} \right], \quad (21)$$

represents the solution of the maximum capacity problem. We note that the equations obtained for the z -matching problem by considering the limit $\beta \rightarrow \infty$ of the BP equations clearly reduce to the equations obtained in [27] for the matching problem when $z = 1$.

Maximum z -matching problem on bipartite network ensembles. –

The BP equations and the energy of the z -matching.

Here we consider the maximum z -matching problem on bipartite network ensembles formed by N users and M towers where the users have degree distribution $\tilde{P}^{(U)}(k)$, and the towers have degree distribution $\tilde{P}^{(T)}(q)$, with $\tilde{P}^{(T)}(q) = 0$ for $q < z$. Note that on a bipartite network the average degree $\langle k \rangle$ of the users and the average degree $\langle q \rangle$ of the towers need to satisfy

$$N \langle k \rangle = M \langle q \rangle. \quad (22)$$

In order to study the energy E of the z -matching problem on these ensembles we denote by $\mathcal{P}^{(U)}(h)$ and $\mathcal{P}^{(T)}(\hat{h})$ the distributions of fields h and \hat{h} , respectively, i.e.

$$\begin{aligned} \mathcal{P}^{(U)}(h) &= w_1 \delta(h, 1) + w_2 \delta(h, -1) + w_3 \delta(h, 0), \\ \mathcal{P}^{(T)}(\hat{h}) &= \hat{w}_1 \delta(\hat{h}, 1) + \hat{w}_2 \delta(\hat{h}, -1) + \hat{w}_3 \delta(\hat{h}, 0), \end{aligned} \quad (23)$$

where $w_1 + w_2 + w_3 = 1$ and $\hat{w}_1 + \hat{w}_2 + \hat{w}_3 = 1$. Therefore, w_1, w_2, w_3 indicate the probability that the cavity fields h coming from users are equal to $1, -1, 0$ respectively, and $\hat{w}_1, \hat{w}_2, \hat{w}_3$ indicate the probability that the cavity fields \hat{h} coming from towers are equal to $1, -1, 0$, respectively.

Using the BP Eqs. (20) derived in the zero temperature limit, we can derive the equations satisfied by the probabilities $\{w_1, w_2, w_3\}$ in a bipartite network ensemble. We have

$$\begin{aligned} w_1 &= \sum_k \frac{k \tilde{P}^{(U)}(k)}{\langle k \rangle} \hat{w}_2^{k-1}, \\ w_2 &= 1 - \sum_k \frac{k \tilde{P}^{(U)}(k)}{\langle k \rangle} (1 - \hat{w}_1)^{k-1}, \\ w_3 &= 1 - w_1 - w_2. \end{aligned} \quad (24)$$

This is consistent with the previous discussion. Namely, a node of degree k sets the outgoing field to 1 if all remaining $k - 1$ neighbours set corresponding incoming fields to -1 ; it sets the outgoing field to -1 if at least one of the remaining $k - 1$ neighbours sets the incoming field to 1 ; otherwise, it sets it to 0 . Similarly, we can derive the equations satisfied by probabilities $\{\hat{w}_1, \hat{w}_2, \hat{w}_3\}$ in the bipartite network ensemble,

$$\begin{aligned} \hat{w}_1 &= \sum_q \frac{q \tilde{P}^{(T)}(q)}{\langle q \rangle} \sum_{p=q-z}^{q-1} \binom{q-1}{p} w_2^p (1 - w_2)^{q-1-p}, \\ \hat{w}_2 &= 1 - \sum_q \frac{q \tilde{P}^{(T)}(q)}{\langle q \rangle} \sum_{p=0}^{z-1} \binom{q-1}{p} w_1^p (1 - w_1)^{q-1-p}, \\ \hat{w}_3 &= 1 - \hat{w}_1 - \hat{w}_2. \end{aligned} \quad (25)$$

Again, as previously discussed, a tower of degree $q \geq z$ sets the outgoing field to $+1$ if at least $q - z$ of remaining $q - 1$ neighbours set corresponding incoming fields to -1 ; it sets the field to -1 if at least z neighbours set

the corresponding fields to +1; otherwise, it sets the field to 0. Finally, the energy E of the maximum z -matching Eq. (21) can be expressed in terms of the probabilities $\{w_1, w_2, w_3, \hat{w}_1, \hat{w}_2, \hat{w}_3\}$ as

$$\begin{aligned}
 E = & N \sum_k \tilde{P}^{(U)}(k) \left[\hat{w}_2^k - \left(1 - (1 - \hat{w}_1)^k \right) \right] \\
 & - M \sum_q \tilde{P}^{(T)}(q) \left[z \sum_{p=z}^q \binom{q}{p} w_1^p (1 - w_1)^{q-p} \right. \\
 & + \sum_{p_1=0}^{z-1} \sum_{p_3=z-p_1}^{q-p_1} \frac{q!}{p_1! p_3! (q - p_1 - p_3)!} p_1 w_1^{p_1} w_3^{p_3} w_2^{q-p_1-p_3} \\
 & \left. + \sum_{p_1=0}^{z-1} \sum_{p_3=0}^{z-p_1-1} \frac{q! (2p_1 + p_3 - z)}{p_1! p_3! (q - p_1 - p_3)!} w_1^{p_1} w_3^{p_3} w_2^{q-p_1-p_3} \right] \\
 & + N \langle k \rangle [w_1 (1 - \hat{w}_2) + \hat{w}_1 (1 - w_2)]. \quad (26)
 \end{aligned}$$

Therefore, the phase diagram of the z -matching problem can be drawn by solving Eqs. (24) and Eqs. (25) and calculating the energy E given by Eq. (26) on this solution as a function of the structural properties of the bipartite network ensemble. Finally we note that the equations obtained here for the z -matching problem defined on bipartite network ensembles reduce to the equations obtained in [27] for the matching problem in the limit case $z = 1$.

Stability condition. The solutions of the BP Eqs. (24) and (25) should be physical, i.e. they should correspond to values of the capacity

$$0 \leq C \leq \min(zM, N). \quad (27)$$

Moreover, they should be dynamically stable. In order to characterise the stability of a given solution we calculate the Jacobian matrix J of the system of equations for the probabilities $\{w_1, w_2, w_3, \hat{w}_1, \hat{w}_2, \hat{w}_3\}$ including Eqs. (24) and Eqs. (25) which is

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & G'_{1,k}(\hat{w}_2) & 0 \\ 0 & 0 & 0 & G'_{1,k}(1 - \hat{w}_1) & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & A(w_2) & 0 & 0 & 0 & 0 \\ B(w_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix},$$

where

$$G'_{1,k}(x) = \sum_k \frac{k(k-1)}{\langle k \rangle} \tilde{P}^{(U)}(k) x^{k-2}, \quad (28)$$

and

$$\begin{aligned}
 A(w_2) &= \sum_q \frac{q \tilde{P}^{(T)}(q)}{\langle q \rangle} \sum_{p=0}^{z-1} \binom{q-1}{p} H_A(\{w_m\}), \\
 B(w_1) &= \sum_q \frac{q \tilde{P}^{(T)}(q)}{\langle q \rangle} \sum_{p=0}^{z-1} \binom{q-1}{p} H_B(\{w_m\}),
 \end{aligned}$$

with

$$\begin{aligned}
 H_A(\{w_m\}) &= [(q-1-p) - (q-1)w_2] w_2^{q-2-p} (1-w_2)^{p-1}, \\
 H_B(\{w_m\}) &= [(q-1)w_1 - p] w_1^{p-1} (1-w_1)^{q-2-p},
 \end{aligned}$$

and in the derivation of $A(w_2)$ we use

$$\sum_{p=q-z}^{q-1} \binom{q-1}{p} w_2^p (1-w_2)^{q-1-p} = \sum_{p=0}^{z-1} \binom{q-1}{p} w_2^{q-1-p} (1-w_2)^p.$$

A given solution of the system of Eqs. (24) and Eqs. (25) is stable if and only if eigenvalues of the Jacobian J are all less than one. In this way we obtain the stability conditions

$$\begin{aligned}
 B(w_1) G'_{1,k}(\hat{w}_2) &< 1, \\
 A(w_2) G'_{1,k}(1 - \hat{w}_1) &< 1. \quad (29)
 \end{aligned}$$

The trivial solution $w_1 = w_2 = \hat{w}_1 = \hat{w}_2 = 0$ and $w_3 = \hat{w}_3 = 1$ deserves a special consideration. This solution corresponds to $E = 0$, i.e. capacity $C = (N + zM)/2$ (recall Eq. (4)). It immediately follows that this solution is physical, i.e. it satisfies Eq. (27) only for

$$N = zM, \quad (30)$$

in which case it corresponds to full capacity

$$C = N = zM. \quad (31)$$

From the study of the BP Eqs. (24) and (25) we observe that these equations admit the trivial solution $w_1 = w_2 = \hat{w}_1 = \hat{w}_2 = 0$ and $w_3 = \hat{w}_3 = 1$ as long as the minimum degree of the nodes is two and the minimum degree of the towers is $z + 1$, i.e. $\tilde{P}^{(U)}(k) = 0$ for $k = 0, 1$ and $\tilde{P}^{(T)}(q) = 0$ for $q \leq z$. However, in order to establish whether this is the solution of the maximum z -matching problem, we need to investigate its stability.

In particular, if we study the stability conditions Eqs. (29) of the trivial solution $w_1 = w_2 = \hat{w}_1 = \hat{w}_2 = 0$ and $w_3 = \hat{w}_3 = 1$, we obtain

$$\begin{aligned}
 \frac{\langle q(q-1) \rangle}{\langle q \rangle} \frac{2\tilde{P}^{(U)}(2)}{\langle k \rangle} &< 1, \\
 \frac{(z+1)z\tilde{P}^{(T)}(z+1)}{\langle q \rangle} \frac{\langle k(k-1) \rangle}{\langle k \rangle} &< 1. \quad (32)
 \end{aligned}$$

Therefore, the instability of the trivial solution on a bipartite network ensemble with given degree distribution is driven by the fraction of users of degree two and the fraction of towers of degree $z + 1$. In particular when the minimum degree of the nodes is greater than two, i.e. $\tilde{P}^{(U)}(1) = \tilde{P}^{(U)}(2) = 0$ and the minimum degree of the towers is greater than $z + 1$, i.e. $\tilde{P}^{(T)}(q) = 0$ for $q \leq z + 1$, as long as $N = zM$ we get that the trivial solution is stable independently of the other properties of the degree distributions of the nodes and of the towers. This generalises the relation found in matching of simple networks [27], in matching of directed networks [6] and on generalised matching in multilayer networks [7].

z-matching in network ensembles. Let us discuss here few examples of the phase diagram of the maximum z -matching on bipartite networks ensembles. Let us start with a simple example of a regular bipartite network in which the degree distributions are given by

$$\begin{aligned}\tilde{P}^{(U)}(k) &= \delta(k, \bar{k}), \\ \tilde{P}^{(T)}(q) &= \delta(q, \bar{q}),\end{aligned}\quad (33)$$

with $\bar{k} > 0$ and $\bar{q} \geq z$ where $\delta(x, y)$ denotes the Kronecker delta. For these networks we clearly have $\bar{k} = \langle k \rangle$ and $\bar{q} = \langle q \rangle$, therefore it follows that Eq. (22) reduces to

$$\bar{k}N = \bar{q}M. \quad (34)$$

In order to characterise the z -matching on this network ensemble, we solve the Eqs. (24) together with Eqs. (25) for the probabilities $\{w_1, w_2, w_3, \hat{w}_1, \hat{w}_2, \hat{w}_3\}$ and we evaluate the energy E using Eq. (26) and hence the capacity C using Eq. (4).

As a function of the values of \bar{k} and \bar{q} we have the following regimes:

- (1) If $z\bar{k} > \bar{q}$, or equivalently $zM > N$ the solution is $w_2 = \hat{w}_1 = 1$ and $w_1 = w_3 = \hat{w}_2 = \hat{w}_3 = 0$ and corresponding to energy $E = -N + zM$ and capacity $C = N$.
- (2) If $z\bar{k} = \bar{q}$ or equivalently $zM = N$ the solution is $w_3 = \hat{w}_3 = 1$ and $w_1 = w_2 = \hat{w}_1 = \hat{w}_2 = 0$ for $\bar{k} > 1$ and $w_1 = \hat{w}_1 = 1$, $w_2 = w_3 = \hat{w}_2 = \hat{w}_3 = 0$ for $\bar{k} = 1$. Both solutions correspond to energy $E = 0$ and capacity $C = N$.
- (3) If $z\bar{k} < \bar{q}$ or equivalently $zM < N$ the solution is $w_1 = \hat{w}_2 = 1$ and $w_2 = w_3 = \hat{w}_1 = \hat{w}_3 = 0$ corresponding to energy $E = N - zM$ and capacity $C = zM$.

As a second example of bipartite network ensemble we consider the bipartite network in which the degree distributions of the towers and the nodes are Poisson distributed according to the distribution

$$\begin{aligned}\tilde{P}^{(U)}(k) &\sim \frac{a^k}{k!} e^{-a} \quad \text{with } 0 \leq k \leq M, \\ \tilde{P}^{(T)}(q) &\sim \frac{b^{q-z}}{(q-z)!} e^{-b} \quad \text{with } z \leq q \leq N,\end{aligned}\quad (35)$$

where a, b are related so that Eq. (22) holds. For infinite networks with the degree distribution of users $\tilde{P}^{(U)}(k)$ and of towers $\tilde{P}^{(T)}(q)$ given by Eq. (35) the full capacity solution is never achieved as we will always have $\tilde{P}^{(U)}(1) > 0$ and $\tilde{P}^{(T)}(z) > 0$. However for finite networks, the fraction of users with degree 1 and the fraction of towers with degree z is likely to be zero if $\tilde{P}^{(U)}(1) < 1/N$ and $\tilde{P}^{(T)}(z) < 1/M$. Thus, in this case it is possible to enter the regime of the trivial solution which guarantees the full capacity. In Figure 1 we show plot the average capacity density C/N versus the average degree $\langle k \rangle$ for bipartite networks having different ratio N/M between the number

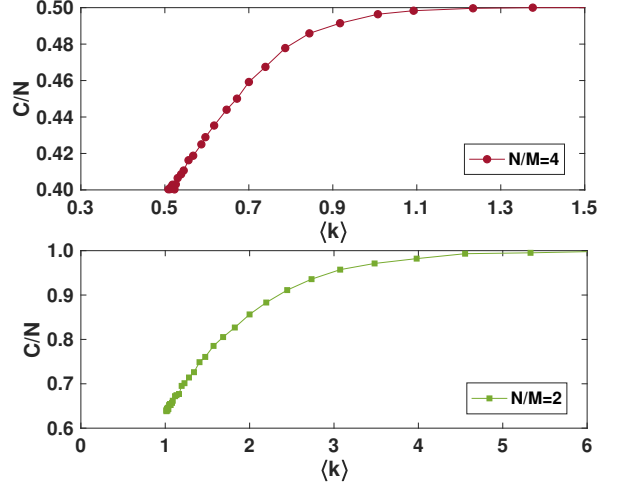


Fig. 1: Capacity density C/N of the z -matching of a bipartite network with Poisson degree distribution of the nodes and of the towers given by Eqs. (35) is plotted versus $\langle k \rangle$ for $z = 2$ at constant value of N/M . The results are obtained using the message passing algorithm averaged over 70 bipartite networks realisations.

of towers and the number of users. We see that for sufficiently high average degree $\langle k \rangle$ the full capacity solution can be achieved. In particular we observe that as the average degree increases the z -matching problem converge to the full capacity solution $C/N = zM/N$ for $zM \leq N$. The results are obtained averaging the results obtained using the message passing algorithm on single realizations of the bipartite networks with degree distributions of users $\tilde{P}^{(U)}(k)$ and of towers $\tilde{P}^{(T)}(q)$ given by Eq. (35) over 70 network realizations.

Conclusions. – In this paper we have analysed the many-to-one z -matching problem using a statistical mechanics approach. This problem is inspired by a wireless network scenario where a set of users needs to be matched to a set of towers providing the wireless connection. While a user can be connected at most with one tower, a tower can serve up to z users. Here we have used the Belief Propagation algorithm in the zero temperature limit to characterise the bipartite network capacity, i.e. the fraction of matched users which is a good proxy for the efficiency of the communication in the network. The phase-diagram of the z -matching problem has been derived for different bipartite network ensembles with given degree distribution.

As the matching problem has recently been related to the controllability of the network, in the future we plan to explore whether also the z -matching problem can be related to the dynamics defined on bipartite networks.

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REFERENCES

- [1] BARABÁSI, A.L., *Network Science* (Cambridge University Press, Cambridge) 2016.
- [2] NEWMAN, M.E.J. , *Networks: An Introduction* (Oxford University Press, Oxford) 2010.
- [3] LATORA, V., NICOSIA, V. AND RUSSO G., *Complex networks: principles, methods and applications* (Cambridge University Press, Cambridge) 2017.
- [4] BIANCONI, G., *Multilayer Networks: Structure and Function* (Oxford University Press, Oxford) 2018.
- [5] LIU, Y.Y., SLOTINE, J.J. and BARABÁSI, A.L., *Nature*, **473** (2011) 167.
- [6] MENICHETTI, G., DALL’ASTA, L. and BIANCONI, G., *Phys. Rev. Lett.*, **113** (2014) 078701.
- [7] MENICHETTI, G., DALL’ASTA, L. and BIANCONI, G., *Sci. Rep.*, **6** (2016) 20706.
- [8] KARRER, B., NEWMAN, M. E. J. and ZDEBOROVÁ L., *Phys. Rev. Lett.*, **113** (2014) 208702.
- [9] BIANCONI, G. , *Phys. Rev. E*, **97** (2018) 022314.
- [10] BOCCALETTI, S., BIANCONI, G., CRIADO, R., DEL GENIO, C., GOMEZ-GARDENES, J., ROMANCE, M., SENDINA-NADAL, I., WANG, Z., ZANIN, M., *Phys. Rep.* , **544** (2014) 1.
- [11] BIANCONI, G., AND DOROGVTSEV, S. N., *Phys. Rev. E*, **89** (2014) 062814.
- [12] CELLAI, D., DOROGVTSEV, S. N. and BIANCONI, G. , *Phys. Rev. E*, **94** (2016) 032301.
- [13] RADICCHI, F. and BIANCONI, G., *Phys. Rev. X*, **7** (2017) 011013.
- [14] OSAT, S., FAQEEH, A. and RADICCHI, F., *Nature Comm.*, **8** (2017) 1540.
- [15] ALTARELLI, F., BRAUNSTEIN, A., DALL’ASTA, L., LAGE-CASTELLANOS, A. and ZECCHINA, R., *Phys. Rev. Lett.*, **112** (2014) 118701.
- [16] ALTARELLI, F., BRAUNSTEIN, A., DALL’ASTA, L., WAKELING, J.R. and ZECCHINA, R., *Phys. Rev. X*, **4** (2014) 021024.
- [17] GLEESON, J. P. and PORTER, M. A., *Complex Spreading Phenomena in Social Systems*, edited by LEHMANN, S. and AHN, Y.-Y. (Springer, Cham) 2018, p. 81.
- [18] LOKHOV, A. Y., and SAAD, D., *Proc. Nat. Aca. Sci.*, **114** (2017) E8138.
- [19] MEZARD, M. and MONTANARI, A., *Information, physics, and computation* (Oxford University Press, Oxford) 2009.
- [20] HARTMANN, A.K. and WEIGT, M., *Phase transitions in combinatorial optimization problems: basics, algorithms and statistical mechanics* (John Wiley & Sons) 2006.
- [21] KARP, R.M. and SIPSER, *Proceedings of the Twenty-second Annual Symposium on Foundations of Computer Science* (IEEE) 1981, p. 364-375.
- [22] ARONSON, J., FRIEZE, A., and PITTEL, B. G., *Random Structures Algorithms*, **12** (1998) 111.
- [23] GALE, D. and SHAPLEY, L.S., *The American Mathematical Monthly*, **69** (1962) 9.
- [24] BORDENAVE, C., LELARGE M. and SALEZ J., *Probability Theory and Related Fields*, **1** (2013) 183.
- [25] MÉZARD, M., PARISI, G. and VIRASORO, M., *Spin glass theory and beyond* (World Scientific) 1987.
- [26] MÉZARD, M. and PARISI, G., *EPL (Europhysics Letters)*, **2** (1986) 913.
- [27] MÉZARD, M. and PARISI, G., *JSTAT*, **5** (2006) P05003.
- [28] BAYATI, M., BORGS, C., CHAYES, J. and ZECCHINA, R., *JSTAT*, **6** (2008) L06001.
- [29] Zhou, H.-J. and Ou-Yang Z.-C. Maximum matching on random graphs. arXiv:cond-mat/0309348 (2003).
- [30] ZHAO, J.H. and ZHOU, H. J., *Phys. Rev. E*, **99** (2019) 012317.
- [31] CHAKRABORTI, A., CHALLET, D., CHATTERJEE, A., MARSILI, M., ZHANG, Y.C. and CHAKRABARTI, B.K., *Phys. Rep.*, **552** (2015) 1.
- [32] OMÉRO, M.J., DZIERZAWA, M., MARSILI, M. and ZHANG, Y.C., *Journal de Physique I*, **7** (1997) 1723.
- [33] CALDARELLI, G. and CAPOCCI, A., *Physica A*, **300** (2001) 325.
- [34] GU, Y., SAAD, W., BENNIS, M., DEBBAH, M. and HAN, Z., *Physica A*, **53** (2015) 52.
- [35] KARTUN-GILES, A., JAYAPRAKASAM, S. and KIM, S., *IEEE Communications Letters*, **22** (2018) 1216.